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Comparison of Methods of Solving Transportation Problems (TP) and Resolving the Associated Variations

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Abstract: Transportation Problem is a Linear Programming application to physical distribution of goods and services from various origins to several destinations such that the cost of transportation is minimal. In this study, five different methods were employed to solve transportation problems arising from unequal demand and supply of goods and variations. The methods considered in terms of North West Corner Rule, Least Cost Method, Vogel's Approximation Method, Row Minima Method and Column Minima Method were compared. Necessary and sufficient condition for the existence of a feasible solution to the transportation problem was initiated and established. Unbalanced transportation problems were resolved using Vogel's Approximation Method (VAM) and Modified Distribution (MODI) methods. The five methods compared produced different results with VAM generating the least transportation cost and better solution. The least value of the transportation costs obtained by the five methods is VAM with the most economical initial feasible solution. It was also established that, out of m + n constraint equations, only m + n-1 equations are linearly independent. With the MODI method, economic values were generated for the dual variables, u_{is} and v_{js} associated with the source and demand points respectively.

Keywords: Transportation Problems, Origins, Destinations, Unbalanced Transportation Problem, Optimal Solution, Optimality Test

1. Introduction

Transportation problem is a special subclass of Linear Programming Problem (LPP) in which the main objective is to transport various quantities of a single homogeneous commodity from certain sources (origins) to destination (demand) points at a minimum cost. In order to achieve the said objective, the amount and location of available supplies and the quantities demanded, as well as transportation costs of one unit of commodity from various origins to various destinations are known in advance. It is also assumed that availability and requirements of various centres are finite with limited resources, cost of transportation is linear and sources or jobs are expressed in terms of only one kind of unit [12, 1].

In an attempt to proffer solutions to transportation problems, several methods have been presented, developed and subsequently extended by various researchers at different time [8, 2, 6, 3].

However, this study compares various methods of solving transportation problems, and also reviews the associated variations in terms of unbalanced demand and supply; degeneracy with respect to its resolution at the initial stage and subsequent or intermediate iterations and prohibited transportation routes using real life instances.

1.1. Definitions of Terms

In this section, definitions of certain terminologies used in TP are given as follows:

a. Feasible solution: A feasible solution to a Transportation Problem is a set of non-negative allocations ($x_{ij} \geq 0$) which satisfies rim (row and column) restrictions.

b. Basic feasible solution: A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to $m+n-1$, where m is the number of rows and n the number of columns in a transportation table.
c. Degenerate basic feasible solution: This is a basic feasible solution that contains less than $m+n-1$ non negative allocations.-
d. Non degenerate basic solution: A basic feasible solution to a ($n \times m$) TP is said to be a non degenerate basic feasible solution if it contains exactly $m+n-1$ nonnegative allocations in independent positions.
e. Independent positions: Allocations in occupied cells are said to be in independent positions if it is impossible to form a closed path.
f. Closed path: A closed path is formed by allowing horizontal and vertical lines such that the corner cells are occupied.
g. Optimal solution: A feasible solution (not necessarily basic) which minimizes the total transportation cost is said to be optimal.
h. Triangular basis: A Transportation Problem has a triangular basis if the system of equations $AX = b$ is represented in terms of basic variables only, and non basic variables are set to zero in the system [5, 4].

1.2. The Transportation Problem

The Transportation Problem is the standard name given to a class of problems in which the transportation is required. The general parameters of transportation are as follows:

(i) Resources: The resources are those elements that can be transported from certain origins to demand points. For example, discrete resources include goods, machines, tools, people, cargo; continuous resources include energy, liquids, and money.
(ii) Locations: The locations are points of supply and recollection, depots, nodes, railway stations, bus stations, loading port, seaports, airports, refueling depots, or school and so on.
(iii) Transportation modes: The transportation modes are the form of transporting some resources to locations. The transportation modes use water, space, air, road, rail, and cable. The form of transport has different infrastructures, capacities, times, activities, and regulations. For example, transportation modes a include ship, aircraft, truck, train, pipeline, motorcycle, and others [9, 14].

1.3. The Structure of Transportation Problem

A Transportation Problem with m origins of supplies S_1, S_2, …, S_m with a_i units of supplies(i = 1, 2, …, m) and n demand points D_1, D_2, …, D_n with b_j units of requirements (j = 1, 2, …, n) is represented using network and tabular structures as follows:

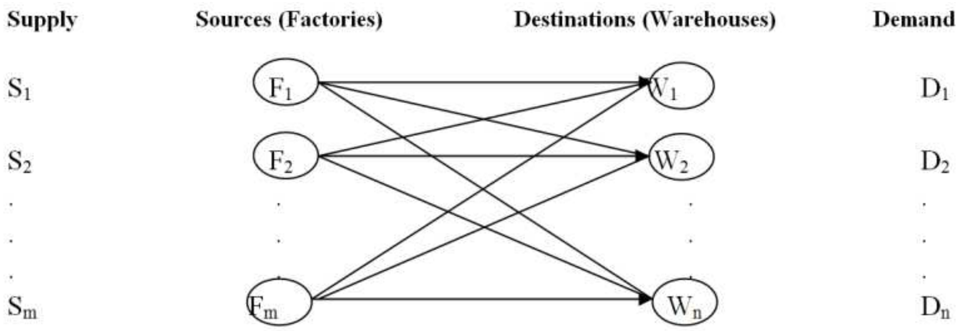

Figure 1. *Network Representation of Transportation Problem.*

Table 1. *Tabular Representation of Transportation Problem.*

	Destination						
		1	**2**	**3**	**…**	**n**	**Supply**
	1	C_{11} x_{11}	C_{12} x_{12}	C_{13} x_{13}		C_{1n} x_{1n}	a_1
	2	C_{21} x_{21}	C_{22} x_{22}	C_{23} x_{23}		C_{2n} x_{2n}	a_2
Origin	3	C_{31} x_{31}	C_{32} x_{32}	C_{33} x_{33}		C_{3n} x_{1n}	a_3
	. . .						. . .
	M	C_{n1} x_{m1}	C_{n2} x_{m2}	C_{n3} x_{m3}		C_{mn} x_{mn}	a_m
	Demand	b_1	b_2	b_3	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

1.4. Formulation of Transportation Model

Consider a transportation problem with m sources and n destinations. Let x_{ij} be the quantity transported from origin i to destination j; C_{ij} is the corresponding transportation cost; a_i is the number of supply units available at source $i\,(i=1,2,...,m)$; b_j is the number of demands units required at destination j ($j=1,2,...,n$) and the total cost $C=\sum_i\sum_j C_{ij}x_{ij}$.

The assumptions of the model are given as:

$$\sum_{i=1}^{m} x_{ij} = b_j, j=1,2,...n \tag{1}$$

(demand requirements)

$$\sum_{i=1}^{m} x_{ij} = a_j, i=1,2,...m \tag{2}$$

(supply constraints)

$$x_{ij} \geq 0 \quad \forall\, i,j \tag{3}$$

(non-negativity restrictions)

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{4}$$

(rim condition)

The Linear Programming Problem (LPP) model of the Transportation Problem (TP) is given as

$$\text{Minimize } Z=\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \tag{5}$$

Subject to

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j=1,2,...n \tag{6}$$

$$\sum_{i=1}^{m} x_{ij} = a_j, \quad i=1,2,...m \tag{7}$$

$$x_{ij} \geq 0 \quad \forall\, i,j \tag{8}$$

The TP is balanced if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{9}$$

Theorem 1: Existence of Feasible Solution

A necessary and sufficient condition for the existence of a feasible solution to a transportation problem is the rim condition which states that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (total supply = total demand).

Proof:

a. Necessary condition is established.

Suppose there is a feasible solution to the transportation problem. Then,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \text{ and } \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} b_j \tag{10}$$

$$\Leftrightarrow \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{11}$$

b. Sufficient Condition is also established

Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = k$. Suppose $\lambda_i \neq 0$ is any real number such that $x_{ij}=\lambda_i b_j$ for all i and j, then λ_i is given as

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \lambda_i b_j = \lambda_i \sum_{j=1}^{n} b_j = k\lambda_i,$$

where $\lambda_i = \frac{1}{k}$

$$\sum_{j=1}^{n} x_{ij} = \frac{a_i}{k} \tag{12}$$

Therefore,

$$x_{ij} = \lambda_i b_j = \frac{a_i b_j}{k}, \text{for all } i \text{ and } j \tag{13}$$

Hence, a feasible solution exists since $a_i > 0,\, b_j > 0,$ for all i and j, $x_{ij} > 0$.

Theorem 2: Existence of Basic Feasible Solution

The number of basic variables in an $m \times n$ transportation table is $m+n-1$

Proof:

Given an m x n transportation table with m origins and n demand points From Theorem 1, out of m + n -1 constraint equation, any one of them may be redundant. This can be eliminated so that the remaining m + n -1 form a linearly independent set.

To establish this, sum up equation (7) and subtract from the sum the first n-1 column equation to obtain the last column equation:

$$\sum_{i=1}^{m}\sum_{j=}^{n}x_{ij}-\sum_{j=1}^{n-1}\sum_{i=1}^{m}x_{ij}=\sum_{i=1}^{m}a_i\sum_{j=1}^{n-1}b_j \tag{14}$$

$$\sum_{i=1}^{m}\sum_{j=}^{n}x_{ij}-\left(\sum_{j=1}^{n-1}\sum_{i=1}^{m}x_{ij}-\sum_{i=1}^{m}x_{in}-\right)=\sum_{i=1}^{m}a_i-\left(\sum_{j=1}^{n-1}b_j-b_n\right), \tag{15}$$

where $\sum_{i=1}^{m}x_{in}=b_n$ since $\sum_{i=1}^{m}a_i=\sum_{j=1}^{n}b_j$.

2. Methodology

Methods of Solving Transportation Problems

In this study, five methods of finding initial basic feasible solutions are considered and compared. An optimality test is also carried out only on the basic feasible solutions with following conditions satisfied:

(a) The number of allocations is m + n -1, where m is the number of rows and n is the number of columns.
(b) The allocations in (a) are in independent positions.

The Transportation Algorithm

The transportation algorithm is the simplex method specialized to the format of the transportation matrix. It involves the following steps [11].

1. Formulate the problem and arrange the data in the matrix form;
2. Obtain an initial basic feasible solution;
3. Test the initial solution for optimality;
4. Improve the solution for optimality; and
5. Update the solution by repeating steps 3 and 4 until an optimal solution is reached.

Finding Initial Basic Feasible Solution to Transportation Problems.

Different methods of obtaining transportation problems are discussed with the assumption that the transportation table is blank at the inception.

2.1. North-West Corner Rule (NWCR)

The NWCR does not take into account the cost of transportation on any route. The steps are:

a. Starting with the cell in the upper left corner of the transportation matrix, allocation is made to

X_{11} as many units as possible without violating the constraints [X_{11}= min(a_1,b_1)].

b. If b_1> a_1, move down vertically to the second row and make second allocation of magnitude

X_{21} = min(a_2, b_1-X_{11}) in the cell (2,1).

If b1 < a1, move one cell horizontally to the second column and make second allocation of magnitude X_{12} = min(a_1, X_{11}- b_1) in the cell (1,2).

If b_1 = a_1, there is a tie for the second allocation and second allocation of magnitude

X_{12} = min(a_1-a_1, b_1) = 0 in the cell (1,2) or X_{21} = min(a_2, b_1-b_1) =0 in cell (2,1) is made.

c. Repeat steps a and b moving down towards the lower right corner of the transportation matrix until the rim requirements are satisfied.

2.2. Least Cost Method (LCM)

This method takes into account the minimum unit cost of transportation for obtaining the initial solution and the steps are summarized as follows:

a. The smallest cost in the cost matrix of the transportation table is obtained. Let it be C_{ij} and allocate X_{ij}= min (a_i,b_j) in the cell (i,j).
b. If X_{ij} = a_i, cross off the i[th] row of the transportation table and decrease b_j by a_i. Then go to step c.
c. Steps a and b are repeated for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minimum.

2.3. Vogel's Approximation Method (VAM)

The VAM is otherwise known as penalty or regret method. In this method, an allocation is made on the basis of the opportunity (extra) cost that would have been incurred if the allocation in certain cells with minimum unit transportation cost were missed. Thus, allocations are made in such a way that the penalty cost is minimized. An initial solution obtained by this method is nearer to an optimal solution or it is optimal solution itself [7]. The steps are summarized:

a. The penalty cost is calculated by obtaining the difference between the smallest and the next smallest costs in each row and column.
b. Among the penalties as found in step a, the maximum penalty is chosen. If there is a tie, one ischosen arbitrarily.
c. In the selected row or column as by step b, the cell having the least cost is found. Allocate as much as possible without violating the constraints.
d. The row or column which is fully exhausted is deleted. Again, the row and column penalties are computed for the reduced transportation matrix and then go to step b. The procedure is repeated until all demands are satisfied.

2.4. The Row Minima Method (RMM)

The minimum transportation cost for RMM can be obtained through these steps:

a. Allocate as much as possible in the lowest cost cell of the first row until the capacity of the first row is exhausted or the requirement at j^{th} distribution centre is satisfied or both.

(i) If the capacity of the first row is completely exhausted, cross off the first row and proceed to second row.

(ii) If the requirement at j^{th} destination is satisfied, cross off the j^{th} column and reconsider the first row with the remaining capacity.

(iii) If the capacity of first row as well as the requirements at j^{th} destination are completely satisfied, make a zero allocation in the second lowest cost cell of the first row. Cross off the row as well as as the j^{th} column and move down the second row.

b. Continue with the process for the resulting transportation table until the rim conditions are satisfied.

2.5. Column Minima Method (CMM)

The minimum transportation cost for CMM can be obtained through these steps:

a. Allocate as much as possible in the lowest cost cell of the first column so that the demand of the first distribution centre is satisfied or the capacity of the i^{th} source is exhausted or both.

(i) If the requirement of the first distribution centre is satisfied, cross out the first column and move right to the second column.

(ii) If the capacity of i^{th} source is satisfied, cross out i^{th} row and reconsider the first column with the remaining requirement.

(iii) If the requirements of the first destination, as well as the capacity of the i^{th} source are completely satisfied, make a zero allocation in the second lowest cost cell of the first column.

Cross out the column as well as the i^{th} row and move right to the second column.

b. Continue with the process for the resulting reduced transportation table until all the rim conditions are satisfied.

2.6. Optimal Solution to Transportation Problem

Sequel to the determination of initial basic solutions to transportation problems, an optimum solution is obtained provided that such allocations have exactly m+n-1 non negative independent allocations. The optimality test is performed in this study using Modified Distribution Method (MODI).

Let $u_1, u_2, u_3, ..., u_n$ and $v_1, v_2, ..., v_n$ be the dual variables associated with the source and demand points constraints respectively. The net evaluations are determined using the characteristics of dual and primal problems.

Theorem 3

If there is a feasible solution consisting of m + n -1 independent allocations, and if u and v satisfy $c_{rs} = u_r + v_s$, for each occupied cell (r,s), then the evaluation d_{ij} corresponding to each empty cell(i,j) is given by

$$d_{ij} = c_{ij} - (u_i - v_j) \tag{16}$$

Proof: The transportation problem is to find $x_{ij} \geq 0$ in order to minimize

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}c_{ij} \tag{17}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, i = 1,2,...m; \tag{18}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, j = 1,2,...n \tag{19}$$

The restrictions (18) and (19) are rewritten as

$$0 = a_i - \sum_{j=1}^{n} x_{ij}, \quad i = 1,2,...,m \tag{20}$$

$$0 = b_j - \sum_{i-1}^{m} x_{ij}, j = 1,2,...,n \tag{21}$$

Any multiple of the equations (20) and (21) is added to equation (17) to eliminate the basic variable. Multiplying $u_i (i = 1,2,...,m), v_j (j = 1,2,...,n)$ respectively, a new equation is generated as:

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}(c_{ij}) + \sum_{i=1}^{m} u_i \left[a_i - \sum_{j=1}^{n} x_{ij} \right] + \sum_{i=1}^{n} v_j \left[b_j - \sum_{i=1}^{m} x_{ij} \right] \tag{22}$$

$$= \sum_{i-1}^{m}\sum_{j=1}^{n} x_{ij}(c_{ij} - (u_i + v_j)) + \sum_{i=1}^{m} u_i a_i + 1\sum_{j=1}^{n} v_j b_j \tag{23}$$

If $c_{rs} = u_r + v_s$, for each basic cell, the net evaluations must vanish for the basic variables. Also, if there are m + n-1 dual equations in m + n dual unknowns, and arbitrary value is assigned to one of the unknowns, u_i and v_j, then the remaining m + n -1 can be uniquely solved.

Setting $u_i = 0$, the remaining values are obtained by addition and subtraction.

In order to prove that $d_{ij} = c_{ij} – (u_i + v_j)$, an empty cell (i,j) which is connected to occupied cells by a closed path is established using Table 2 as shown below:

Table 2. *Closed Path of Cells.*

	D_1	…	D_j	…	D_s	…	D_n	Supply
S_1								a_1
:								:
.								.
S_i			+1 c_{ij}	$\leftarrow$	-1 c_{is}			a_i
:			-1 $\downarrow$		$\downarrow$ +1			:
.								.
S_r			c_{rj}	$\rightarrow$	c_{rs}			a_r
S_m								
Demand	b_1	…	b_j		b_s		b_n	a_m

Allocate +1unit to the empty cell (i,j) and add -1 unit to occupied cell (r,j) to balance the total requirements of D_j. The total amount available from origin S_r is balanced by adding +1unit to occupied cell (r,s), which causes D_s to be unbalanced. Column D_s is balanced by adding -1 unit to the occupied cell (i,s).

The procedure gives the cost difference d_{ij} (empty cell evaluation of cell (i,j) between the new solution and the original solution. Thus,

$$d_{ij} = c_{ij} - c_{rj} + c_{rs} - c_{is} \tag{24}$$

Since $c_{rs} = u_r + v_s$ for all occupied cell (r,j), (r,s) and (i,s), then

$$dij = c_{ij} –(u_r + v_j)+ (u_r+ v_s) – (u_i + v_s)$$

$$= c_{ij} – (u_i+v_j)$$

Similarly, the result for all empty cell (i,j) to occupied cell is generalized.

2.7. Test for Optimality Using Modified Distribution (MODI) Method

The check for optimality is significant in terms of feasibility of the solution to transportation problem once an initial solution is obtained. The Modified Distribution (MODI) or u-v method (is employed to compute the associated opportunity cost associated with each unoccupied cell (unused route) and then improving the current solution leading to an optimal solution. An occupied cell with the largest negative opportunity cost is selected to include in the new set of transportation allocations. The value indicates the per unit cost reduction that can be achieved by making appropriate allocation in the occupied cell. The outgoing cell from the current solution is the occupied cell where allocation will become zero as allocation is made in the occupied cell with largest negative opportunity cost [13, 15]. The process continues until the current solution is optimal. The steps for optimality test are given as:

1. Find the initial basic feasible solution of the given TP by using any of the five methods earlier discussed.
2. Find out a set of numbers u_i and v_j for each row and column satisfying $u_i + v_j = c_{ij}$ for each cell occupied. To start with, assign a number zero to any row or column having maximum number of allocations. If there is a tie, choose one arbitrarily.
3. For each unoccupied cell, find the sum of u_i and v_j (indicated at the bottom-left corner of that cell).
4. Find out for each unoccupied cell the net evaluation value $d_{ij} = c_{ij} – (u_i+v_j)$ which is indicated at the bottom right corner of that cell. This step gives the following optimality conditions:
   (i) If all $d_{ij} > 0$, the solution is optimum and unique solution exists;
   (ii) If $d_{ij} \geq 0$, the solution is optimum, but an alternative solution exists;
   (iii) If at least one $d_{ij} < 0$, the solution is not optimum, go to step 5.
5. Select the unoccupied cell having the most negative d_{ij}. From the cell, draw a closed path by drawing horizontal and vertical lines with corner cells occupied. Assign +ve and –ve signs alternately and find the minimum allocation from the cell having –ve sign. This allocation should be added to the allocations having +ve sign and subtracted from the allocations having –ve signs.
6. Step 5 yields a better solution of making one (or more) occupied cells unoccupied and one unoccupied cell as occupied. Repeat steps 2-5 until an optimum basic feasible solution is obtained.

2.8. Variations in Transportation Problems

Unbalanced Transportation Problem

Transportation Problems (TP) are solved with the assumption that the total supply is equal to the total demand. If they are unequal, the TP is unbalanced it is resolved by considering the following cases: (i) If the total supply is more than total demand, a dummy column with zero cost is introduced in the cost matrix. The excess supply is entered as a rim requirement for the dummy destination; (ii) If the total demand is more than the total supply, an additional row is introduced with zero cost in the cost matrix. The excess demand is entered as a rim requirement for this dummy source [13, 10].

2.9. Degeneracy in Transportation Problems

In an m x n transportation table, the number of basic or occupied cells must be m + n -1. The basic solution degenerates if the number of basic cells is less than m + n -1. Degeneracy needs to be removed in order to improve the given solution.

2.9.1. Degeneracy at the Initial Stage

At the initial basic solution, the number of occupied cells may be less than m + n -1. In order to resolve the degeneracy at this stage, allocate very small quantity $\in$ (close to zero) to one or more of the unoccupied cells so that a number of occupied cells become m + n– 1. The quantity does not affect the total transportation cost nor the demand and supply values. In a minimization problem, $\in$ is allocated to unoccupied cells that have the lowest transportation costs while the same quantity is allocated to a cell that has a high payoff value in maximization problem.

2.9.2. Degeneracy at Subsequent Iterations

At any stage while moving towards optimality, two or more occupied cells may become simultaneously unoccupied. To resolve degeneracy during optimality test, allocate the quantity ε to one or more of recently vacated cells so that the number of occupied cell is m+n -1 in the new solution. The small quantity $\in$ is subject to the following conditions:

$(a) \in < x_{ij}, \forall x_{ij} > 0$

$(b) \in +0 = \in$

$(c) x_{ij} \pm \in = x_{ij} \ \forall x_{ij} > 0$

(d) If there are two or more $\in s$ in the solution, i.e $\in$ and $\in'$, then $\in < \in'$ whenever is above $\in'$. If $\in$ and $\in'$ are in the same row, $\in < \in'$ when $\in$ is to the left of $\in'$.

2.9.3. Alternative Optimal Solutions

The existence of alternative optimal solutions is determined by inspecting the opportunity costs, d_{ij} for the unoccupied cells. If $d_{ij} = 0$, for an unoccupied cell in an optimal solution, then an alternative optimal solution exists and it is obtained by bringing such an unoccupied cell in the solution mix without increasing the transport cost.

2.9.4. Prohibited Transportation Routes

Unforeseen scenarios such as road hazards (flood, kidnapping, traffic congestion, etc) may come up and as a result prompt transportation of goods from certain origins to different destinations. This can be resolved by assigning a very large cost, M or ∞ to such routes.

3. Applications and Discussion of Results

Given the following Transportation Problem, the total transportation costs of the initial feasible solutions for the five methods are determined and compared. The transportation costs in Naira per unit from certain factories in Lagos to certain warehouses in Abuja are displayed below:

Table 3. *Transportation Problem.*

Origins	D_1	D_2	D_2	D_3	Supply
S_1	2	3	11	7	6
S_2	1	0	6	1	1
S_3	5	8	15	9	10
Demand	7	5	3	2	17

3.1. North West Corner Method (NWCM)

From Table 3, the rim conditions are satisfied since $\sum a_i = \sum b_j$ and therefore there exists a feasible solution to the transportation problem.

Step1: The first allocation is made in the cell (1,1) with magnitude x_{11} = min(6,7) =7. This strikes out the first row.

Step 2: In the cell (2,1), the magnitudex_{21} = min(1, 1) =1. This crosses out the first column and second row.

Step 3: In the cell (3,2), the magnitude x_{32} = min (10,5) = 5. This Crosses out the second column and proceed to cell (3,3). Allocate x_{33} = min(5,3) = 3. Step 4: In cell (3,4), allocate x_{34} = min(2,2) =2.

The basic solution is displayed in the table below:

Table 4. *Initial Basic Feasible Solution.*

$\boxed{6}$ 2	3	11	7
$\boxed{1}$ 1	0	6	1
5	$\boxed{5}$ 8	$\boxed{3}$ 15	$\boxed{2}$ 9

Since the solution does not form a loop, it is basic and the initial transportation cost generated by NWCM is

$$z = (2x6)+(1x1)+(8x5)+(15x3)+(9x2) = ₦116$$

3.2. Least Cost Method (LCM)

From Table 3, the lowest cost cell (2,2) and the maximum feasible allocation in cell (2,2) =1.

Since x_{21} meets supply at S_2, cross off row two. Similarly, maximum feasible allocations for cells (1,1), (3,1), (3,2), (3,4), (3,3) are x_{11}= 6, x_{31} = 1, x_{32} = 4, x_{34} = 2, x_{33} = 3 respectively.

Table 5. *Initial Basic Feasible Solution.*

$\boxed{6}$ 2	3	11	7
1	$\boxed{1}$ 0	6	1
$\boxed{1}$ 5	$\boxed{4}$ 8	$\boxed{3}$ 15	$\boxed{2}$ 9

The transportation cost generated by LCM is z = (2 x 6) + (0 x 1) + (5 x 1) + (8 x 4) + (15 x 3) + (9 x 2) = ₦112

3.3. Vogel's Approximation Method (VAM)

Taking all the allocations (1^{st} – 6^{th}) and following steps 1-5, the initial basic feasible solution using VAM is obtained as:

Table 6. *Initial Basic Feasible Solution.*

$\boxed{1}$ 2	$\boxed{5}$ 3	11	7
1	0	6	$\boxed{1}$ 1
$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9

Thus, $x_{11} = 2$, $x_{12} = 3$, $x_{24} = 1$, $x_{31} = 5$, $x_{33} = 15$, $x_{34} = 9$

The transportation cost generated by VAM is

z = (2 x 1) + (3 x 5) + (1 x 1) + (5 x 6) + (15 x 3) +(9 x 1) = ₦102

3.4. Row Minima Method (RMM)

With all the allocations (1^{st} – 6^{th}), the initial basic feasible solution to the transportation problem is as shown:

Table 7. *Initial Basic Feasible Solution.*

$\boxed{6}$ 2	3	11	7
1	$\boxed{1}$ 0	6	1
$\boxed{1}$ 5	$\boxed{4}$ 8	$\boxed{3}$ 15	$\boxed{2}$ 9

The transportation cost generated by RMM is

z = (2 x 6) + (0 x 1) + (5 x 1) + (8 x 4) + (15 x 3) + (9 x 2) = ₦112

3.5. Column Minima Method (CMM)

With all the allocations (1^{st} – 5^{th}), the initial basic feasible solution to the transportation problem is as shown below:

Table 8. *Initial Basic Feasible Solution.*

$\boxed{6}$ 2	3	11	7
$\boxed{1}$ 1	0	6	1
$\boxed{1}$ 5	$\boxed{5}$ 8	$\boxed{3}$ 15	$\boxed{2}$ 9

The transportation cost generated by CMM is

z = (2 x 6) + (1 x 1) + (5 x 1) + (8 x 5) + (15 x 3) + (9 x 2) = ₦116

3.6. Optimality Test by MODI Method

A Transportation Problem in Nigeria is considered using VAM method:

Table 9. *Transportation Problem.*

Origins	D_1	D_2	D_2	D_3	Supply
S_1	21	16	25	13	11
S_2	17	18	14	23	13
S_3	32	27	18	41	19
Demand	6	10	12	15	43

Since total demand is equal to total supply, the TP is balanced. Therefore, an initial basic feasible solution exists in Table 7 using VAM algorithm.

Table 10. *Initial Basic Feasible Solution.*

21	16	25	$\boxed{11}$ 13
$\boxed{6}$ 17	$\boxed{18}$	$\boxed{3}$ 14	$\boxed{4}$ 23
32	$\boxed{10}$ 17	$\boxed{9}$ 18	41

The total transportation cost is

z = (13 x 11) + (17 x 6) + (14 x 3) + (23 x 4) + (17 x 10) + (18 x 9) =₦711

To find if the initial basic feasible solution is optimal, calculate u_i and v_j using c_{ij}.

Starting with $u_2 = 0$ (the row with highest number of allocations),

$$c_{21} = u_2 + v_1 \Rightarrow v_1 = 17$$

$$c_{23} = u_2 + v_3 \Rightarrow v_3 = 14$$

$$c_{24} = u_2 + v_4 \Rightarrow v_4 = 23$$

$$c_{14} = u_1 + v_4 \Rightarrow u_1 = -10$$

$$c_{33} = u_3 + v_3 \Rightarrow u_3 = 4$$

$$c_{32} = u_3 + v_2 \Rightarrow v_2 = 13$$

The net evaluation $d_{ij} = c_{ij} – (u_i + v_j)$ is computed for each of the unoccupied cell

$$d_{11} = c_{11} – (u_1+v_1)=14$$

$$d_{12} = c_{12} –(u_1 +v_2)= 13$$

$$d_{13} = c_{13} – (u_1+v_3)= 21$$

$$d_{22} = c_{22} – (u_2 + v_2)= 5$$

$$d_{31} = c_{31} – (u_3 + v_1)= 11$$

$$d_{34} = c_{34} – (u_3 + v_4)= 14$$

All the net evaluations are greater than zero. Hence the initial basic feasible solution is optimal and transportation cost is also unique (minimum cost = 711).

g. Resolving Unbalanced TP

The Unbalanced TP is resolved by considering unit transportation costs, demands and supplies below:

Table 11. *Unbalanced Transportation Problem.*

Origin	D_1	D_2	D_3	D_4	Supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	4	7	70
Demand	85	35	50	45	

Since Total Demand = 215 ≠ Total Supply = 195, a dummy origin 0 and cost zero are Introduced with supply equal to 20 units as shown below:

Table 12. *Balanced Transportation Problem.*

Origin	D_1	D_2	D_3	D_4	Supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	4	7	70
O_4	0	0	0	0	20
Demand	85	35	50	45	215

Table 13. *Initial Basic Feasible Solution by VAM Method.*

Origin	D_1	D_2	D_3	D_4	Supply	u_i
O_1	$\boxed{6}$ 65	$\boxed{1}$ 5	$\boxed{9}$	$\boxed{3}$	70	$u_1 = 0$
O_2	$\boxed{11}$	$\boxed{5}$ 30	$\boxed{2}$ 25	$\boxed{8}$	55	$u_2 = 4$
O_3	$\boxed{10}$	$\boxed{12}$	$\boxed{4}$ 25	$\boxed{7}$ 45	70	$u_3 = 6$
O_4	$\boxed{0}$ 20	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	20	$u_4 = -6$
Demand	85	35	50	45	215	
	$v_1 = 6$	$v_2 = 1$	$v_3 = -2$	$v_4 = 1$		

There are 7 independent non negative allocations which equal to m + n – 1. Thus, the solution is non degenerate and the total transportation cost is

z =(65 x 6) + (5 x 1) + (30 x 5) + (25 x 2) x (25 x 4)
x (45 x 7) + (20 x 0) = ₦1010

To find the optimal solution, steps in MODI methods are applied to obtain the optimum solution below:

Table 14. *Initial Table for Optimal Solution.*

	D_1	D_2	D_3	D_4	Supply
O_1	$\boxed{-6}$ 65	$\boxed{1+}$ 5	$\boxed{9}$	$\boxed{3}$	70
O_2	$\boxed{11}$	5 - 30	$\boxed{2+}$ + 25	$\boxed{8}$	55
O_3	$\boxed{10}$ +	$\boxed{12}$	4 - 25	$\boxed{7}$ 45	70
O_4	$\boxed{0}$ 20	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	20
Demand	85	35	50	45	215

Cell (3,1) is allocated to min(65,30,25) = 25

The modified allocation is given below:

Table 15. *Modified Allocation Table.*

	D_1	D_2	D_3	D_4	Supply	u_i
O_1	$\boxed{6}$ 40	$\boxed{1}$ 30	$\boxed{9}$	$\boxed{3}$	70	$u_1 = 6$
O_2	$\boxed{11}$	$\boxed{5}$ 5	$\boxed{2}$ 30	$\boxed{8}$	55	$u_2 = 10$
O_3	$\boxed{10}$ 25	$\boxed{12}$	4	$\boxed{7}$ 45	70	$u_3 = 10$
O_4	$\boxed{0}$ 20	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	20	$u_4 = 0$
Demand	85	35	50	45	215	
v_j	$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -3$		

The number of independent allocations is m + n -1 =7. Also, u_i, v_j are computed to find d_{ij} and using occupied cells. For the unoccupied cells, the $d_{ij} = c_{ij} - (u_i + v_j)$ are:

$$d_{13} = 11,\ d_{14} = 0,\ d_{21} = 1,\ d_{24} = 1,$$

$$d_{32} = 7,\ d_{31} = 2,\ d_{42} = 5,\ d_{43} = 8\ d_{44} = 3$$

Since all d_{ijs} are greater or equal to zero, there exists an optimality of allocations. The optimal allocations are $x_{11} = 40$, $x_{12} = 30$, $x_{22} = 5$, $x_{23} = 50$, $x_{31} = 25$, $x_{34} = 45$, $x_{41} = 20$ and the optimal cost is (40 x 6) + (30 x 1) + (5 x5) + (50 x 2) + (25 x 10) + (45 x 7) + (20 x 0) = 960.

4. Findings

(a) Comparing the five methods of transportation cost, the Vogel Approximation Method (VAM) has the least generated transportation cost. This is followed by both Least Cost Method (LCM) and Row Minimum Method (RMM) with the highest transportation cost accrued to both North West Corner Rule (NWCR) and Column Minimal Method (CMM).

(b) Optimality test carried out by MODI method resulted to computation of opportunity cost associated with each unoccupied cell and then improved the current solution leading to an optimal solution. Negative opportunity costs obtained indicate the unit cost reduction achieved by raising the shipment allocation in the unoccupied cell from its present level of zero.

(c) The u_{is} and v_{js} have economic interpretations in the sense that u_i values measure the comparative advantage of additional unit of supply or shadow price of available supply at point i (location rent). Similarly, the v_j values measure the comparative advantage of an additional unit of commodity demanded at demand point j (market price).

5. Conclusion

The five methods used in this study have produced different results except for NWCR and CMM with the same result and also LCM and RMM with similar result. The Vogel Approximation Method (VAM) produces the least transportation cost, hence a better solution. The MODI method employed also compares the relative advantage of alternative allocations for all the unoccupied cells simultaneously.

References

[1] Abdallah, A. H. and Mohammad, A. A. (2012). Solving Transportation Problems using the Best Candidates Methods. *Journal of Computer Science and Engineering, 2 (5),* pp 23-30.

[2] Ahmed, M. M., Khan, A. R., Uddin, M. D. and Ahmed, F. (2016). A New Approach to Solving Transportation Problems. *Open Journal of Optimization, 5 (3),* pp 22-30.

[3] Farzana, S. R. and Safiqul, I. (2020). A Comparative Study of Solving Methods of Transportation Problem in Linear Programming Problem. *British Journal of Mathematics and Computer Science, 35 (5),* pp 45- 57.

[4] Funso, A. (1995). Practical Operational Research for Developing Countries (A Process Frame Work Approach. Panaf Press, Lagos, Nigeria.

[5] Hamdy, A. T. (2017). Operations Research, An Introduction. Pearson Education Limited, 10th ed., England.

[6] Joshi, R. V. (2015). Optimization Techniques for Transportation Problems of Three Variables. *IOSR Journal of Mathematics*, *9 (1),* pp 46-50.

[7] Kalavathy, S. (2013). Operations Research. VIKAS Publishing House, PVT LTD.

[8] Mohammad, H. and Farzana, S. F. (2018). A New Method for Optimal Solutions of Transportation Problems in LPP. *Journal of Mathematics*, *10 (5),* pp 60-75.

[9] Natarajan, A. M., Balasubramani, P. and Tamilarasi, A. (2005). Operations Research. Pearson Education, Singapore.

[10] Ocotlan, D. P., Jorge, A. R., Beatriz, B. L., Alejandro, F. P. and Richardo, A. B. C. (2013). A Survey of Transportation Problems. *Journal of Applied Mathematics, 2014*, pp 1-17.

[11] Richard, B. and Govindasami, N. (1997). Schaum's Outline of Theory and Problems of Operations Research, McGraw Hill Co., Toronto.

[12] Sharma, J. K. (2017). Quantitative Techniques for Managerial Decisions. Macmillan Publishers, India.

[13] Shraddha, M. (2017). Solving Transportation Problems by Various Methods. *International Journal of Mathematics Trends and Technology, 44 (4),* pp 270-275.

[14] Ganesh, A. H., Suresh, M. and Sivakumar, G. (2020). On Solving Fuzzy Transportation Problem based on Distance based Defuzzification Method of Various Fuzzy Quantities using Centroid. Malaya Journal of Matematik, (1), pp 410-426.

[15] Havir, S. K. and Krishna, D. K. (2004). Introductory Operations Research. Springer Science and Business Media LLC.